Generalised Ornstein-Uhlenbeck processes

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Abstract

We solve a physically significant extension of a classic problem in the theory of diffusion, namely the Ornstein-Uhlenbeck process [G. E. Ornstein and L. S. Uhlenbeck, Phys. Rev. **36**, 823, (1930)]. Our generalised Ornstein-Uhlenbeck systems include a force which depends upon the position of the particle, as well as upon time. They exhibit anomalous diffusion at short times, and non-Maxwellian velocity distributions in equilibrium. Two approaches are used. Some statistics are obtained from a closed-form expression for the propagator of the Fokker-Planck equation for the case where the particle is initially at rest. In the general case we use spectral decomposition of a Fokker-Planck equation, employing nonlinear creation and annihilation operators to generate the spectrum which consists of two staggered ladders.

1 Introduction

This paper introduces a physically important extension of a classic problem in the theory of diffusion, namely the Ornstein-Uhlenbeck process [1]. Our results are obtained by spectral decomposition of a linear operator. The spectrum of this operator consists of two ladders of eigenvalues with respectively odd and even parity. The ladders of eigenvalues are staggered, that is the odd-even step is different from the even-odd step (see Fig. 1). The corresponding eigenfunctions are generated by a raising operator. A concise account of our work on these staggered ladder spectra appeared earlier [2]. In the following we show how the results summarised in [2] were obtained. We also derive new results, not included in our earlier report: a closed-form solution for example, and the generalisation of our previous results to a continuous family of diffusion processes.

1.1 The Ornstein-Uhlenbeck process

Before we discuss our extension of the Ornstein-Uhlenbeck process, we describe its usual form [1]. This considers a particle of momentum p subjected to a rapidly fluctuating random force $f(t)$ and subject to a drag force $-\gamma p$, so that the equation of motion is

$$\dot{p} = -\gamma p + f(t)\,. \tag{1}$$

The random force has statistics $\langle f(t)\rangle = 0$, $\langle f(t)f(t')\rangle = C(t-t')$ (angular brackets denote ensemble averages throughout). If the correlation time τ of $f(t)$ is sufficiently short ($\gamma\tau \ll 1$), the equation of motion may be approximated by a Langevin equation:

$$\mathrm{d}p = -\gamma p \mathrm{d}t + \mathrm{d}w\,, \tag{2}$$

where the Brownian increment $\mathrm{d}w$ has statistics $\langle \mathrm{d}w\rangle = 0$ and $\langle \mathrm{d}w^2\rangle = 2D_0\mathrm{d}t$. The diffusion constant is

$$D_0 = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t\ \langle f(t)f(0)\rangle\,. \tag{3}$$

This problem is discussed in many textbooks (for example [3]).

1.2 Generalised Ornstein-Uhlenbeck processes

Our extension arises when the force depends upon position as well as time. We consider the case where the fluctuations of the force on the particle are mainly a consequence of the spatial, rather the temporal, fluctuations of the force $f(x,t)$. A consequence of this difference is that the impulse δw supplied to the particle in a short time δt depends upon the momentum of the particle. If the particle is at position x_0 at time t_0, this impulse is

$$\delta w = \int_{t_0}^{t_0+\delta t} \mathrm{d}t\, f\big(x_0 + p(t-t_0)/m, t\big) + O(\delta t^2)\,. \tag{4}$$

In particular, the impulse approaches zero as the speed $|p|/m$ of the particle increases, because the motion of the particle effects an average over the spatial fluctuations of the force. This can be seen clearly by considering the second moment of δw. We assume that the force $f(x,t)$ has the following statistics

$$\langle f(x,t)\rangle = 0\,, \quad \langle f(x,t)f(x',t')\rangle = C(x-x',t-t')\,. \tag{5}$$

The spatial and temporal correlation scales of the random force $f(x,t)$ are ξ and τ respectively. We consider the case where (for most of the time) the momentum of the particle is large compared to $p_0 = m\xi/\tau$, then the force experienced by the particle decorrelates more rapidly than the force experienced by a stationary particle. If δt is large compared to τ but small compared to $1/\gamma$, we can estimate the variance of the impulse $\langle \delta w^2 \rangle = 2D(p)\delta t$ as follows (due to translational invariance, we consider without loss of generality a particle which starts from position $x = 0$ at time $t = 0$)

$$\begin{aligned} \langle \delta w^2 \rangle &= \int_0^{\delta t} \mathrm{d}t_1 \int_0^{\delta t} \mathrm{d}t_2 \ \big\langle f(pt_1/m, t_1) f(pt_2/m, t_2) \big\rangle \\ &= \delta t \int_{-\infty}^{\infty} \mathrm{d}t \ C(pt/m, t) + O(\tau^2) \, . \end{aligned} \tag{6}$$

We define the momentum diffusion constant by writing $\langle \delta w^2 \rangle = 2D(p)\delta t + O(\delta t^2)$, and find

$$D(p) = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}t \ C(pt/m, t) \ . \tag{7}$$

When $p \ll p_0$ we recover $D(p) = D_0$. When $p \gg p_0$, we can approximate (7) to obtain

$$D(p) = \frac{D_1 p_0}{|p|} + O(p^{-2}) \ , \ D_1 = \frac{m}{2p_0} \int_{-\infty}^{\infty} \mathrm{d}X \ C(X, 0) \ . \tag{8}$$

When the force is the gradient of a potential $V(x,t)$ with a correlation function having continuous derivatives, we find that D_1 is zero. This case is discussed in section 8, where it is shown that $D(p) \sim |p|^{-3}$ provided the correlation function of $V(x,t)$ is sufficiently differentiable. Another variation, also discussed in section 8, arises when the correlation function of the force exhibits a discontinuity at $t = 0$ (as when the potential $V(x,t)$ is itself generated by an Ornstein-Uhlenbeck process). In this case $D(p) \sim |p|^{-2}$, and other exponents are also possible. We therefore consider a general situation where $D(p) \sim |p|^{-\zeta}$ and give exact results for case

$$D(p) = D_\zeta \, (p_0/|p|)^\zeta \tag{9}$$

with $\zeta \geq 0$. We analyse the dynamics by solving a Fokker-Planck equation which determines the probability density for the Langevin process in which the momentum has diffusion constant given by (9). We discuss the form of this Fokker-Planck equation in section 2; the remainder of this introduction will set our work in context with earlier research on related topics.

1.3 Earlier work

The motion of a damped particle subjected to a force fluctuating in both space and time was first studied by Deutsch [4], who addressed an entirely different aspect of the problem. Deutsch considered to case where the momentum of the particle remains small compared to p_0, and posed the question of whether particles aggregate. He discovered that there is a phase transition between coalescing and non-coalescing trajectories. (Two of the authors of the present paper subsequently solved Deutsch's one-dimensional model exactly [5], and

results for two and three spatial dimensions are discussed in [6, 7]). All of these papers only considered cases where $p \ll p_0$.

Sturrock [8] analysed the motion of a particle subjected to a spatially varying force field without damping. He introduced the concept of a momentum diffusion constant which varies as a function of the momentum: that is, he considered the same problem as is addressed in the present paper, but in the limit of damping constant $\gamma = 0$. Subsequently Golubovic, Feng, and Zeng [9] identified the importance of the relation $D(p) \sim |p|^{-3}$ (in the case of a potential force), and discussed the nature of the Fokker-Planck equation and its solution in the case where $\gamma = 0$. It was argued that the particle exhibits anomalous diffusion and solution for the propagator of the Fokker-Planck equation with initial value $p = 0$ was proposed. Later Rosenbluth [10] pointed to an error in the evaluation of this propagator. The results of [8, 9, 10] were applied to the stochastic acceleration [11] of particles in plasmas, and subsequent contributions have concentrated on refining models for the calculation of $D(p)$ (see, for example, [12, 13]).

In the following we analyse the problem with the damping term, proportional to γ, included. Surprisingly, we find that this more general problem is more tractable: we are able, for example, to obtain precise results concerning the problems considered in [9, 10] by taking, in our solutions, the limit $\gamma \to 0$.

There is a large literature devoted to the motion of particles advected in random velocity fields (corresponding to the large-γ limit of the model we study). In the case where the velocity field is independent of time, sub-diffusive motion is typically found [14, 15]. The advection of tracers in a turbulent fluid is described by models with rapidly fluctuating velocity fields [16]. In our problem the inertia of the particles plays an important role. Particles suspended in a turbulent fluid can show surprising clustering properties when inertia effects are significant. These were first proposed by Maxey [17]; the current state of knowledge is summarised in [7]. In cases where the random force results from motion of the surrounding fluid, it is not possible for the condition $p \gg p_0$ to be realised [7].

We remark that a brief summary of many of the results of this paper was already published [2]. The closed-form solution of section 3, the WKB analysis and most of the results for general values of ζ were not discussed in this earlier work.

1.4 Description of our results and outline of this paper

In order to simplify the presentation, we describe in detail only our results for the case $\zeta = 1$, corresponding to a generic random force. Corresponding expressions for general values of ζ are obtained using the same method, and we quote the most important results for general values of ζ in section 8 at the end of the paper.

In section 2, the Fokker-Planck equation for the generalised Ornstein-Uhlenbeck processes is described. In section 3 we briefly discuss a particular closed-form solution, which enables us to determine the steady state momentum distribution (which is non-Maxwellian) and some statistics, such as the time evolution of the variance of the momentum. The results of section 3 are not sufficient to enable all statistics to be calculated, and in the general case we obtain statistics via a spectral decomposition of the Fokker-Planck equation. Section 4 discusses this spectral decomposition. We transform the Fokker-Planck operator into a Hermitean operator and determine the eigenvalues and eigenvectors of this 'Hamiltonian' operator by generating them using a new type of raising

and lowering operators, which are nonlinear second-order differential operators. We show that the resulting spectrum is a ladder spectrum, consisting of separate ladders for the odd and even parity states. These are staggered: the odd-even separation differs from even-odd. Section 5 contains calculations of the matrix elements needed for computing correlation functions and expectation values. In section 6 we summarise our results on diffusion and anomalous diffusion for generic random forcing.

Section 7 discusses a technical issue concerning our evaluation of the spectrum. When the index of the eigenvalue is large, it is possible to apply standard WKB approximation methods everywhere except in the vicinity of a singularity of the Hamiltonian. We show that the singularity introduces phase shifts which explain the staggered-ladder structure of the spectrum.

Finally, in section 8 we explain in more detail how other values of ζ can arise and summarise our results for general ζ.

2 Fokker-Planck equations

We consider a particle with equations of motion

$$\dot{x} = p/m \ , \quad \dot{p} = -\gamma p + f(x,t) \tag{10}$$

where the force $f(x,t)$ is random, with statistics given by equation (5). In the limit as the correlation time τ of the force approaches zero, the equation of motion of the momentum may be approximated by a Langevin equation, (2), where the random increment $\mathrm{d}w$ has second moment $\langle \mathrm{d}w^2\rangle = 2D(p)\mathrm{d}t$ with $D(p)$ given by (7). This Langevin equation for the stochastic evolution of $p(t)$ corresponds to a Fokker-Planck equation (generalised diffusion equation) for the probability density of the momentum, $P(p,t)$. Using standard results [3], the Fokker-Planck equation is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial p}\big(v(p)P\big) + \frac{\partial^2}{\partial^2 p}\big(D(p)P\big) \tag{11}$$

where

$$v(p) = \frac{\langle \mathrm{d}p\rangle}{\mathrm{d}t} \ , \quad D(p) = \frac{\langle \mathrm{d}p^2\rangle}{2\mathrm{d}t} \ . \tag{12}$$

Note that we can replace $\mathrm{d}p$ by $\mathrm{d}w$ in the expression for $D(p)$, because the neglected terms are of higher order in $\mathrm{d}t$, and that $D(p)$ has already been obtained in equation (7). In order to determine the correct form of the Fokker-Planck equation it remains to determine $\langle \mathrm{d}p\rangle = -\gamma \mathrm{d}t + \langle \mathrm{d}w\rangle$.

Expanding the impulse (4) about a reference trajectory $x(t) = pt/m$, and using the fact that $\langle f(x,t)\rangle = 0$, we obtain

$$\langle \delta w\rangle \ = \ \frac{1}{m}\int_0^{\delta t}\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3\ \exp[-\gamma(t_2-t_3)]\left\langle \frac{\partial f}{\partial x}(pt_1/m,t_1)f(pt_3/m,t_3)\right\rangle . \tag{13}$$

Note that throughout the three-dimensional region of integration, we have $0 \le t_3 \le t_2 \le t_1 \le \delta t$, and the short correlation time implies that the integrand is negligible unless $|t_1 - t_3| < \tau$. The integrand is therefore significant along a line rather than a

surface, because t_2 must lie between t_1 and t_3. The integral is therefore $O(\delta t)$, rather than $O(\delta t^2)$ which would obtain if the integrand were significant on a surface. We replace the factor $\exp[-\gamma(t_2 - t_3)]$ by unity because $\gamma\tau \ll 1$, and the other factor is negligible when $|t_2 - t_3| > \tau$. The integral over t_2 then gives simply $t_1 - t_3$. Writing $t = t_1 - t_3$, in the limit $\gamma\tau \ll 1$, the result is therefore

$$\langle \delta w \rangle = \frac{1}{2m} \int_{-\infty}^{\infty} \mathrm{d}t\ t \left\langle \frac{\partial f}{\partial x}(0,0) f(pt/m, t) \right\rangle = \delta t \frac{\mathrm{d}}{\mathrm{d}p} D(p)\,. \tag{14}$$

This implies

$$v(p) = \frac{\mathrm{d}}{\mathrm{d}p} D(p)\,. \tag{15}$$

Rosenbluth [10] has pointed out that this relation can also be obtained as a consequence of applying the principle of detailed balance.

With (7) and (15), the following Fokker-Planck equation obtains:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial p}\big(\gamma p + D(p)\frac{\partial}{\partial p}\big) P\,. \tag{16}$$

Sturrock [8] introduced a related Fokker-Planck equation (without the damping term) and also gave an expression for $D(p)$ analogous to equation (7).

In the following we discuss our solution of (16) for the particular case of generic random forcing (corresponding to $\zeta = 1$). Results for other values of ζ are obtained in an analogous fashion. The general case is briefly described in section 8.

3 A particular closed-form solution

In this section we introduce a particular solution of the Fokker-Planck equation (16) with $D(p)$ given by (9). We restrict ourselves to the case of generic random forcing (corresponding to $\zeta = 1$):

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial p}\big(\gamma p + D_1 \frac{p_0}{|p|}\frac{\partial}{\partial p}\big) P\,. \tag{17}$$

Consider the distribution $P(p,t)$ of momentum p for particles initially at rest. It satisfies the initial condition $P(p,0) = \delta(p)$ where $\delta(p)$ is the δ-function. For this particular initial condition, we have found the following closed-form solution of (17):

$$P(p,t) = \frac{1}{2\Gamma(4/3)} \frac{\gamma^{1/3}}{[3p_0 D_1(1-\mathrm{e}^{-3\gamma t})]^{1/3}} \exp\left[-\frac{\gamma |p|^3}{3p_0 D_1 (1-\mathrm{e}^{-3\gamma t})}\right]\,. \tag{18}$$

Eq. (18) determines how the moments of momentum grow for a particle initially at rest:

$$\langle p^{2l}(t)\rangle = \Big(\frac{3D_1}{\gamma}\Big)^{2l/3} \frac{\Gamma\big((2l+1)/3\big)}{\Gamma(1/3)} \big(1-\mathrm{e}^{-3\gamma t}\big)^{2l/3} \tag{19}$$

for positive integers l. This result is consistent with the result obtained in [2] (eq. (8) in that paper). In the limit of small times (19) gives rise to anomalous diffusion

$$\langle p^{2l}(t)\rangle \sim t^{2l/3}\,. \tag{20}$$

At large times ($\gamma t \gg 1$), by contrast, we obtain a stationary non-Maxwellian momentum distribution

$$P_0(p) = \frac{1}{2\Gamma(4/3)} \frac{\gamma^{1/3}}{(3p_0 D_1)^{1/3}} \exp\left[-\gamma|p|^3/(3p_0 D_1)\right]. \tag{21}$$

The particular solution (18) generalises in a natural way to other values of ζ.

However, in order to determine the momentum correlation function and the spatial diffusion properties, the particular solution (18) is not sufficient, the general solution for arbitrary initial condition is required. We have not been able to obtain the general solution to (16) in closed form. Therefore, we determine it using spectral decomposition: we construct the eigenvalues λ_n and eigenfunctions $\psi_n(z)$ of a Hermitian operator $\hat{H}$ corresponding to the Fokker-Planck equation (17). We identify raising and lowering operators $\hat{A}^+$ and $\hat{A}$ which map one eigenfunction to another with respectively two more or two fewer nodes. We use these to obtain the spectrum and eigenfunctions of $\hat{H}$ which in turn allow us to construct the propagator, expectation values and correlation functions. This approach is described in sections 4 to 5.

4 Spectral decomposition

Introducing dimensionless variables ($t' = \gamma t$ and $p = zp_0[D_1/(\gamma p_0^2)]^{1/3}$) we write (17) as

$$\frac{\partial P}{\partial t'} = \frac{\partial}{\partial z}\big(z + \frac{1}{|z|}\frac{\partial}{\partial z}\big)P \equiv \hat{F}P \tag{22}$$

It is convenient to transform the Fokker-Planck operator $\hat{F}$ to a Hermitian form which we shall refer to as the Hamiltonian operator:

$$\hat{H} = P_0^{-1/2}\hat{F}P_0^{1/2} = \frac{1}{2} - \frac{|z|^3}{4} + \frac{\partial}{\partial z}\frac{1}{|z|}\frac{\partial}{\partial z}. \tag{23}$$

Here $P_0(z) \propto \exp(-|z|^3/3)$ is the stationary solution (21) satisfying $\hat{F}P_0 = 0$. We solve the diffusion problem by constructing the eigenfunctions of the Hamiltonian operator. In the following we make use of Dirac notation [18] of quantum mechanics to write the equations in a compact form and to emphasise their structure.

The eigenfunctions of the Fokker-Planck equation (16) are alternately even and odd functions, defined on the interval $(-\infty, \infty)$. The operator $\hat{H}$, describing the limiting case of this Fokker-Planck operator, is singular at $z = 0$. We identify two eigenfunctions of $\hat{H}$ by inspection, $\psi_0^+(z) = \mathcal{C}_0^+ \exp(-|z|^3/6)$ which has eigenvalue $\lambda_0^+ = 0$ and $\psi_0^-(z) = \mathcal{C}_0^- \, z|z| \exp(-|z|^3/6)$ with eigenvalue $\lambda_0^- = -2$. These eigenfunctions are of even and odd parity, respectively (zero and one node, respectively). Our approach to determining the full spectrum is to define a raising operator $\hat{A}^+$ which maps any eigenfunction $\psi_n^\pm(z)$ to its successor with the same parity, $\psi_{n+1}^\pm(z)$, having two additional nodes.

4.1 Algebra of raising and lowering operators

We write

$$\hat{H} = \hat{a}^-|z|^{-1}\hat{a}^+. \tag{24}$$

Here $a^{\pm} = (\partial_z \pm z|z|/2)$. We introduce the operators

$$\hat{A} = \hat{a}^{+}|z|^{-1}\hat{a}^{+} \quad \text{and} \quad \hat{A}^{+} = \hat{a}^{-}|z|^{-1}\hat{a}^{-} \tag{25}$$

as well as

$$\hat{G} = \hat{a}^{+}|z|^{-1}\hat{a}^{-} \,. \tag{26}$$

Note that $\hat{A}^{+}$ is the Hermitian conjugate of $\hat{A}$. The commutator of $\hat{A}$ and $\hat{A}^{+}$ is

$$[\hat{A}, \hat{A}^{+}] = -3(\hat{H} + \hat{G}) \,. \tag{27}$$

Note also that $\hat{H} - \hat{G} = \hat{I}$ (where $\hat{I}$ is the identity operator).

4.1.1 Eigenvalues

It can be verified that

$$[\hat{H}, \hat{A}] = 3\hat{A} \quad \text{and} \quad [\hat{H}, \hat{A}^{+}] = -3\hat{A}^{+} \,. \tag{28}$$

These expressions show that the action of $\hat{A}$ and $\hat{A}^{+}$ on any eigenfunction is to produce another eigenfunction with eigenvalue increased or decreased by three, or else to produce a function which is identically zero. The operator $\hat{A}^{+}$ adds two nodes, and repeated action of $\hat{A}^{+}$ on $\psi_0^{+}(z)$ and $\psi_0^{-}(z)$ therefore exhausts the set of eigenfunctions. Together with $\lambda_0^{+} = 0$ and $\lambda_0^{-} = -2$ this establishes that the spectrum of $\hat{H}$ is (see Fig. 1)

$$\lambda_n^{+} \;=\; -3n \quad \text{and} \quad \lambda_n^{-} = -3n - 2 \quad n = 0, \ldots, \infty \,. \tag{29}$$

4.1.2 Eigenfunctions

We represent the eigenfunctions by of $\hat{H}$ by kets $|\psi_n^{-}\rangle$ and $|\psi_n^{+}\rangle$. The actions of $\hat{A}$ and $\hat{A}^{+}$ are

$$\hat{A}^{+}|\psi_n^{\pm}\rangle \;=\; C_{n+1}^{\pm}|\psi_{n+1}^{\pm}\rangle \quad \text{and} \quad \hat{A}|\psi_n^{\pm}\rangle = C_{n+1}^{\pm}|\psi_{n-1}^{\pm}\rangle \,. \tag{30}$$

The normalisation factor C_{n+1}^{-} is determined as follows:

$$1 = \langle\psi_{n+1}^{-}|\psi_{n+1}^{-}\rangle = (C_{n+1}^{-})^{-2}\langle\psi_n^{-}|\hat{A}\hat{A}^{+}|\psi_n^{-}\rangle = (C_{n+1}^{-})^{-2}\langle\psi_n^{-}|[\hat{A},\hat{A}^{+}] + \hat{A}^{+}\hat{A}|\psi_n^{-}\rangle \,. \tag{31}$$

It follows

$$(C_{n+1}^{-})^2 = \Big(3\,(-2\lambda_n^{-} + 1) + (C_n^{-})^2\Big) \,. \tag{32}$$

By recursion we obtain

$$C_n^{-} = [3n(3n+2)]^{1/2} \,. \tag{33}$$

This determines the normalisation of the states $(\hat{A}^{+})^n|\psi_0^{-}\rangle$

$$|\psi_n^{-}\rangle = N_n^{-}(\hat{A}^{+})^n|\psi_0^{-}\rangle \tag{34}$$

with

$$N_n^{-} = \left(\prod_{k=1}^{n} 3k(3k+2)\right)^{-1/2} N_0^{-} \,. \tag{35}$$

For the positive-parity states we proceed in a similar fashion and obtain

$$C_n^+ = [3n(3n-2)]^{1/2}\,. \tag{36}$$

This implies

$$|\psi_n^+\rangle = N_n^+(\hat{A}^+)^n|\psi_0^+\rangle \tag{37}$$

with

$$N_n^+ = \left(\prod_{k=1}^{n} 3k(3k-2)\right)^{-1/2} N_0^+\,. \tag{38}$$

The operators $\hat{A}^+$ and $\hat{A}$ differ from the usual examples of raising and lowering operators in that they are of second order in $\mathrm{d}/\mathrm{d}z$, whereas other examples of raising and lowering operators are of first order in the derivative. The difference is associated with the fact that the spectrum is a staggered ladder: only states of the same parity have equal spacing, so that the raising and lowering operators must preserve the odd-even parity. This suggests replacing a first-order operator which increases the quantum number (total number of nodes) by one with a second-order operator which increases the quantum number by two, preserving parity.

There is an alternative approach to generating the eigenfunctions of $\hat{H}$. This equation falls into one of the classes considered in [19], and we have written down first-order operators which map one eigenfunction into another. However, these operators are themselves functions of the quantum number n, making the algebra cumbersome. The approach is briefly described in the next section.

4.2 Schrödinger factorisation

Consider the eigenvalue problem

$$\hat{H}|\psi\rangle = \lambda|\psi\rangle \tag{39}$$

with Hamiltonian (23). For $z > 0$ it can be transformed by the variable change $x = z^3$:

$$\left[(3x)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}}{\mathrm{d}x} - x\left(\frac{1}{4}x + \lambda - \frac{1}{2}\right)\right]\psi(x) = 0 \tag{40}$$

by means of the transformation $x = z^3$. Eq. (40) is a Fuchsian linear differential equation with regular singular points of rank less than or equal to two. Eq. (40) can therefore be factorised using a generalised Schrödinger factorisation scheme (see [19] for a review of this method).

Applying this scheme we have obtained raising and lowering operators generating the spectrum (29). The raising operator acting on $\psi_n^\pm$ is given by

$$T_{\pm,n+1} = -3x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{2} - 3(n+\delta_\pm) \tag{41}$$

with $\delta_+ = 1/3$ and $\delta_- = 1$. The lowering operator acting on $\psi_n^\pm$ is

$$\tilde{T}_{\pm,n} = -3x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}x + 3(n+\eta_\pm) \tag{42}$$

with $\eta_+ = 0$ and $\eta_- = 2/3$. Note that the Hermitian conjugates of $T_{\pm,n+1}$ and $\tilde{T}_{\pm,n}$ are

$$\begin{aligned} (T_{\pm,n+1})^+ &= -\tilde{T}_{\pm,n+1} + 3 && (43) \\ (\tilde{T}_{\pm,n})^+ &= -T_{\pm,n} + 3\,. && (44) \end{aligned}$$

The raising and lowering operators satisfy

$$T_{\pm,n+1}|\psi_n^\pm\rangle = C_{n+1}^\pm|\psi_{n+1}^\pm\rangle \quad \text{and} \quad \tilde{T}_{\pm,n}|\psi_n^\pm\rangle = C_n^\pm|\psi_{n-1}^\pm\rangle \tag{45}$$

with $C_n^\pm$ given by (33) and (36), generating the spectrum (29). The operators differ from $\hat{A}$ and $\hat{A}^+$ introduced in section 4 in that they are of first order in $\mathrm{d}/\mathrm{d}x$, and in that they depend on the state they are applied to.

5 Correlation functions and matrix elements

5.1 Correlation functions

The required solutions of the Fokker-Planck equation may be expressed in terms of the propagator $K(y,z,t)$ which is the probability density for the scaled momentum to reach z after time t, starting from y. It satisfies the Fokker-Planck equation $\partial_{t'}K = \hat{F}K$ and can be expressed in terms of the eigenvalues λ_n^σ and eigenfunctions $\phi_n^\sigma(z) = P_0^{-1/2}\psi_n^\sigma(z)$ of $\hat{F}$:

$$K(y,z;t') = \sum_{n\sigma} a_n^\sigma(y)\phi_n^\sigma(z)\exp(\lambda_n^\sigma t') \tag{46}$$

for $t' > 0$. Here y is the initial value and z is the final value of the coordinate. The expansion coefficients $a_n^\sigma(y)$ are determined by the initial condition $K(y,z;0) = \delta(z-y)$, namely $a_n^\sigma(y) = P_0^{-1/2}\psi_n^\sigma(y)$. In terms of the eigenfunctions of $\hat{H}$ we have (for $t' > 0$)

$$K(y,z;t') = \sum_{n\sigma} P_0^{-1/2}(y)\psi_n^\sigma(y)P_0^{1/2}(z)\psi_n^\sigma(z)\exp(\lambda_n^\sigma t')\,. \tag{47}$$

Equilibrium correlation functions of an observable $O(z)$ are given by

$$\langle O(z_0)O(z_{t'})\rangle_{\mathrm{eq.}} = \int_{-\infty}^{\infty}\mathrm{d}z\int_{-\infty}^{\infty}\mathrm{d}y\; O(z)O(y)K(y,z;t')P_0(y)\,. \tag{48}$$

Since $P_0^{1/2}(y) = \psi_0^+(y)$ this corresponds to

$$\langle O(z_0)O(z_{t'})\rangle_{\mathrm{eq.}} = \sum_{n\sigma} |\langle\psi_0^+|\hat{O}|\psi_n^\sigma\rangle|^2\exp(\lambda_n^\sigma t') \tag{49}$$

for $t' > 0$. The momentum correlation function in equilibrium, for instance, is

$$\langle p_t p_0\rangle_{\mathrm{eq.}} = p_0^2\left(\frac{D_1}{\gamma p_0^2}\right)^{2/3}\sum_n \langle\psi_0^+|\hat{z}|\psi_n^-\rangle^2\exp(\lambda_n^- t') \tag{50}$$

for $t' > 0$, which requires the evaluation of matrix elements $\langle\psi_0^+|\hat{z}|\psi_n^-\rangle$.

Consider on the other hand the time-dependence of $\langle x^2(t)\rangle$, with particles initially at rest at the origin. We need to evaluate

$$\langle x^2(t)\rangle = \frac{1}{m^2}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\ \langle p_{t_1} p_{t_2}\rangle\,. \tag{51}$$

In dimensionless variables this corresponds to

$$\langle x^2(t)\rangle \;=\; \frac{1}{\gamma^2}\left(\frac{p_0 D_1}{\gamma}\right)^{2/3}\frac{1}{m^2}\int_0^{t'} \mathrm{d}t_1' \int_0^{t'} \mathrm{d}t_2'\ \langle z_{t_1'} z_{t_2'}\rangle\,. \tag{52}$$

The required correlation function is (assuming $t_2' > t_1' > 0$)

$$\begin{aligned}\langle z_{t_2'} z_{t_1'}\rangle &= \int_{-\infty}^{\infty}\mathrm{d}z_1 \int_{-\infty}^{\infty}\mathrm{d}z_2\ z_1 z_2\, K(z_1,z_2;t_2'-t_1')\,K(0,z_1;t_1')\\ &= \sum_{n,m}\frac{\psi_m^+(0)}{\psi_0^+(0)}\langle\psi_0^+|\hat z|\psi_n^-\rangle\langle\psi_n^-|\hat z|\psi_m^+\rangle \exp[\lambda_n^-(t_2'-t_1')+\lambda_m^+ t_1']\,.\end{aligned} \tag{53}$$

In order to evaluate (53), the ratios of wave-function amplitudes $\psi_m^+(0)/\psi_0^+(0)$ are required in addition to matrix elements of $\hat z$. The matrix elements $\langle\psi_n^-|\hat z|\psi_m^+\rangle$ are determined in section 5.2, while the ratios of eigenfunctions are calculated in section 5.3.

5.2 Matrix elements $\langle\psi_m^+|\hat z|\psi_n^-\rangle$

To evaluate the matrix elements $Z_{mn} = \langle\psi_m^+|\hat z|\psi_n^-\rangle$ we proceed in three steps: we first evaluate Z_{0n}, in a second step, the matrix elements Z_{mn} are related to $Z_{0,n-m}$ for $m \le n$. Third, Z_{mn} is evaluated for $m > n$.

5.2.1 Matrix elements $\langle\psi_0^+|\hat z|\psi_n^-\rangle$

Consider first $Z_{0n} = \langle\psi_0^+|\hat z|\psi_n^-\rangle$. These matrix elements are obtained by recursion. To evaluate

$$Z_{0,n+1} \;=\; \langle\psi_0^+|\hat z\hat A^+|\psi_n^-\rangle/C_{n+1}^- \tag{54}$$

we write $\hat z\hat A^+ = \hat z\hat G + \hat z(\hat A^+ - \hat G) = \hat z(\hat H - \hat I) + \hat z(\hat A^+ - \hat G)$. It follows

$$\langle\psi_0^+|\hat z\hat A^+|\psi_n^-\rangle = (\lambda_n^- - 1)Z_{0n} + \langle\psi_0^+|\hat z(\hat A^+ - \hat G)|\psi_n^-\rangle\,. \tag{55}$$

Using $(\hat A^+ - \hat G) = -\hat z\hat a^-$ and $[\hat z^2, \hat a^-] = -2\hat z$ we obtain

$$\langle\psi_0^+|\hat z\hat A^+|\psi_n^-\rangle = (\lambda_n^- + 1)Z_{0n}\,. \tag{56}$$

This corresponds to the recursion

$$Z_{0n} = (-1)^n\frac{\prod_{k=0}^{n-1}(3k+1)}{\sqrt{\prod_{k=0}^{n-1}3(k+1)(3k+5)}}Z_{00} \tag{57}$$

With $Z_{00} = 3^{-5/12}\sqrt{\pi}/\Gamma(2/3)$ (found by direct evaluation of an integral) we obtain

$$Z_{0n} = (-1)^n\frac{3^{-5/12}}{\sqrt{2\pi}}\sqrt{\Gamma(2/3)}\frac{\Gamma(n+1/3)}{\sqrt{\Gamma(n+1)\Gamma(n+5/3)}}\,. \tag{58}$$

5.2.2 Matrix elements $\langle\psi_m^+|\hat{z}|\psi_n^-\rangle$ for $m \le n$

Consider now the case $m \le n$. Let

$$J_{mn} = \langle\psi_0^+|\hat{A}^m \hat{z}(\hat{A}^+)^n|\psi_0^-\rangle = \langle\psi_0^+|\hat{A}^m[\hat{z},\hat{A}^+](\hat{A}^+)^{n-1}|\psi_0^-\rangle + \langle\psi_0^+|\hat{A}^m\hat{A}^+\hat{z}(\hat{A}^+)^{n-1}|\psi_0^-\rangle\,.$$

We use $[\hat{z},\hat{A}^+] = -(|\hat{z}|^{-1}\hat{a}^- + \hat{a}^-|\hat{z}|^{-1})$ to write

$$\begin{aligned} J_{mn} &= -\langle\psi_0^+|\hat{A}^m\hat{|}z|^{-1}\hat{a}^-(\hat{A}^+)^{n-1}|\psi_0^-\rangle - \langle\psi_0^+|\hat{A}^m\hat{a}^-|\hat{z}|^{-1}(\hat{A}^+)^{n-1}|\psi_0^-\rangle \\ &\quad +\langle\psi_0^+|\hat{A}^m\hat{A}^+\hat{z}(\hat{A}^+)^{n-1}|\psi_0^-\rangle \\ &= J_{mn}^{(1)} + J_{mn}^{(2)} + J_{mn}^{(3)} \end{aligned} \tag{59}$$

and evaluate the three terms separately. The third one gives

$$J_{mn}^{(3)} = \langle\psi_0^+|\hat{A}^m\hat{A}^+\hat{z}(\hat{A}^+)^{n-1}|\psi_0^-\rangle = (C_m^+)^2\langle\psi_0^+|\hat{A}^{m-1}\hat{z}(\hat{A}^+)^{n-1}|\psi_0^-\rangle = (C_m^+)^2 J_{m-1n-1}\,. \tag{60}$$

Consider next the first term: $\hat{A}^m|\hat{z}|^{-1}\hat{a}^-(\hat{A}^+)^{n-1} = \hat{A}^{m-1}\hat{a}^+|\hat{z}|^{-1}\hat{a}^+|\hat{z}|^{-1}\hat{a}^-(\hat{A}^+)^{n-1} = \hat{A}^{m-1}\hat{a}^+|\hat{z}|^{-1}\hat{G}(\hat{A}^+)^{n-1}$, and thus

$$\langle\psi_0^+|\hat{A}^m|\hat{z}|^{-1}\hat{a}^-(\hat{A}^+)^{n-1}|\psi_0^-\rangle = (\lambda_{n-1}^- - 1)\langle\psi_0^+|\hat{A}^{m-1}\hat{a}^+|\hat{z}|^{-1}(\hat{A}^+)^{n-1}|\psi_0^-\rangle\,. \tag{61}$$

Using $\hat{A}^{m-1}\hat{a}^+|\hat{z}|^{-1}(\hat{A}^+)^{n-1} = \hat{A}^{m-1}\hat{a}^+|\hat{z}|^{-1}\hat{a}^-|\hat{z}|^{-1}\hat{a}^-(\hat{A}^+)^{n-2} = \hat{A}^{m-1}\hat{G}|\hat{z}|^{-1}\hat{a}^-\hat{A}^{+n-2}$ we obtain the recursion

$$J_{mn}^{(1)} = (\lambda_{n-1}^- - 1)(\lambda_{m-1}^+ - 1)J_{m-1n-1}^{(1)} = 3n(3m-2)J_{m-1n-1}^{(1)} \tag{62}$$

Now consider $J_{mn}^{(2)}$. Using $\hat{a}^-|\hat{z}|^{-1}(\hat{A}^+)^{n-1} = \hat{A}^+|\hat{z}|^{-1}\hat{a}^-\hat{A}^{+n-2}$ it follows

$$J_{mn}^{(2)} = (C_m^+)^2 J_{m-1n-1}^{(1)}\,. \tag{63}$$

This implies

$$J_{mn}^{(2)} = \frac{(C_m^+)^2}{(\lambda_{n-1}^- - 1)(\lambda_{m-1}^+ - 1)} J_{mn}^{(1)} = \frac{m}{n} J_{mn}^{(1)} \tag{64}$$

Note also that $J_{0n}^{(3)} = 0$, as well as $J_{0n}^{(2)} = 0$. This gives $J_{0n}^{(1)} = J_{0n}$ [consistent with (64)]. We obtain

$$J_{mn}^{(1)} = \prod_{k=1}^{m} 3(n-m+k)(3k-2)J_{0n-m} \tag{65}$$

which results in

$$J_{mn}^{(1)} = (-1)^{m+n}\frac{\Gamma(2/3)3^{m+n+5/6}}{6\pi}\frac{\Gamma(n+1)\Gamma(m+1/3)\Gamma(n-m+1/3)}{\Gamma(n-m+1)}\,. \tag{66}$$

Eq. (66) allows us to write down an inhomogeneous recursion for J_{mn}:

$$J_{mn} = 3m(3m-2)J_{m-1n-1} + (1+m/n)J_{mn}^{(1)} \tag{67}$$

where the inhomogeneous term is given by (66). Iterating this recursion we obtain

$$J_{mn} = \sum_{l=0}^{m} \left(\prod_{k=l+1}^{m} 3k(3k-2) \right) \left(1+\frac{l}{n-m+l}\right) J^{(1)}_{ln-m+l} \tag{68}$$

which results in

$$J_{mn} = (-1)^{m-n} \frac{3^{m+n+1}\Gamma(2/3)^2}{4\pi^2} (m+n+1) \frac{\Gamma(n+1)\Gamma(m+1/3)\Gamma(n-m+1/3)}{\Gamma(n-m+2)} J_{00}\,. \tag{69}$$

This determines J_{mn} for $n \geq m$.

5.2.3 Matrix elements $\langle\psi^+_m|\hat{z}|\psi^-_n\rangle$ for $m > n$

For $m > n$ we use instead

$$J_{mn} = \langle\psi^+_0|\hat{A}^m \hat{z} \hat{A}^{+n}|\psi^-_0\rangle = \langle\psi^+_0|\hat{A}^{m-1}[\hat{A},\hat{z}]\hat{A}^{+n}|\psi^-_0\rangle + \langle\psi^+_0|\hat{A}^{m-1}\hat{z}\hat{A}(\hat{A}^+)^n|\psi^-_0\rangle \tag{70}$$

and proceed as before. We find that $J_{mn} = 0$ for $m > n+1$. For $m = n+1$ we obtain

$$\begin{aligned} J_{n+1n} &= \langle\psi^+_0|\hat{A}^n|\hat{z}|^{-1}\hat{a}^+(\hat{A}^+)^n|\psi^-_0\rangle + \langle\psi^+_0|\hat{A}^n\hat{a}^+|\hat{z}|^{-1}(\hat{A}^+)^n|\psi^-_0\rangle + \langle\psi^+_0|\hat{A}^n|\hat{z}|\hat{A}(\hat{A}^+)^n|\psi^-_0\rangle \\ &= \tilde{J}^{(1)}_{n+1n} + \tilde{J}^{(2)}_{n+1n} + \tilde{J}^{(3)}_{n+1n} \end{aligned} \tag{71}$$

Consider first $\tilde{J}^{(3)}_{n+1n}$:

$$\tilde{J}^{(3)}_{n+1n} = (C^-_n)^2 \tilde{J}^{(3)}_{nn-1} = 3n(3n+2)\tilde{J}^{(3)}_{nn-1}\,. \tag{72}$$

Second, using $\hat{A}^n|\hat{z}|^{-1}\hat{a}^+(\hat{A}^+)^n = \hat{A}^{n-1}\hat{a}^+|\hat{z}|^{-1}\hat{A}(\hat{A}^+)^n$ we determine $\tilde{J}^{(1)}_{n+1n}$:

$$\tilde{J}^{(1)}_{n+1n} = (C^-_n)^2 \tilde{J}^{(2)}_{nn-1} = 3n(3n+2)\tilde{J}^{(2)}_{nn-1}\,. \tag{73}$$

Third,

$$\tilde{J}^{(2)}_{n+1n} = (\lambda^+_n - 1)(\lambda^-_{n-1} - 1)\tilde{J}^{(2)}_{nn-1} = (3n+1)3n\, J^{(2)}_{nn-1}\,. \tag{74}$$

We deduce that

$$\tilde{J}^{(1)}_{n+1n} = \frac{3n+2}{3n+1}\tilde{J}^{(2)}_{n+1n} \tag{75}$$

and obtain the recursion

$$J_{n+1n} = \tilde{J}^{(1)}_{n+1n} + \tilde{J}^{(2)}_{n+1n} + \tilde{J}^{(3)}_{n+1n} = 3n(3n+2)J_{nn-1} + \left(1 + \frac{3n+2}{3n+1}\right)\tilde{J}^{(2)}_{n+1n}\,. \tag{76}$$

Iterating (74) we obtain

$$\tilde{J}^{(2)}_{n+1n} = \left(\prod_{k=1}^{n} 3k(3k+1) \right) \tilde{J}^{(2)}_{10} = \frac{3\sqrt{3}}{2\pi} 9^n \Gamma(2/3)\Gamma(n+1)\Gamma(n+4/3) J_{00} \tag{77}$$

and thus from (76)

$$J_{n+1n} = \frac{\sqrt{3}}{2\pi} 9^n \Gamma(2/3)\Gamma(2+n)\Gamma(n+4/3)\, J_{00}\,. \tag{78}$$

Comparing this result to (69) we find that (69) gives the correct result for $m = n+1$, although it was derived assuming $m \le n$. Normalising to obtain Z_{mn} our final result is

$$Z_{mn} = (-1)^{m-n}\frac{3^{5/6}}{6\pi}(m+n+1)\Gamma(2/3)\frac{\sqrt{\Gamma(n+1)\Gamma(m+1/3)}}{\sqrt{\Gamma(m+1)\Gamma(n+5/3)}}\frac{\Gamma(n-m+1/3)}{\Gamma(n-m+2)} \tag{79}$$

for $n \ge m-1$ and zero otherwise.

5.3 Ratios of eigenfunctions

In this section we show how to evaluate $\psi_n^+(0)/\psi_0^+(0)$. For $z > 0$, the eigenfunctions $\psi_n^+(z)$ are of the form $N_n^+ g_n(z)\exp(-z^3/6)$, where $g_n(z)$ is polynomial in z^3, of the form

$$g_n(z) = g_n^{(0)} + g_n^{(1)} z^3 + \ldots \tag{80}$$

We determine how $\hat{A}^+$ and $\hat{H}$ act on these polynomials. To this end we define

$$\begin{aligned} \hat{A}'^+ &= \mathrm{e}^{z^3/6}\hat{A}^+\mathrm{e}^{-z^3/6} = (\partial_z - z^2)z^{-1}(\partial_z - z^2) && (81)\\ \hat{H}' &= \mathrm{e}^{z^3/6}\hat{H}\mathrm{e}^{-z^3/6} = (\partial_z - z^2)z^{-1}\partial_z\,. && (82) \end{aligned}$$

This implies

$$\hat{A}'^+ - \hat{H}' = -(\partial_z - z^2)z = z^3 - z\partial_z - 1 \tag{83}$$

and thus

$$(\hat{A}'^+ - \hat{H}')g_n = -g_n^{(0)} + O(z^3)\,. \tag{84}$$

Using $\lambda_n^+ = -3n$ we obtain

$$H' g_n = -3n g_n^{(0)} + O(z^3) \tag{85}$$

from the eigenvalue equation. Taking (84) and (85) together we have

$$\hat{A}'^+ g_n = -(3n+1)g_n^{(0)} + O(z^3)\,. \tag{86}$$

Eq. (86) implies

$$g_{n+1}^{(0)} = -(3n+1)g_n^{(0)}\,. \tag{87}$$

With (30) it follows

$$\psi_{n+1}^+(0) = N_{n+1}^+ g_{n+1}^{(0)} = -(3n+1)N_{n+1}^+/N_n^+\psi_n^+(0) = -\sqrt{\frac{3n+1}{3(n+1)}}\psi_n^+(0)\,. \tag{88}$$

Our final result for the ratio of wave-function amplitudes is therefore

$$\psi_n^+(0)/\psi_0^+(0) = (-1)^n\sqrt{\frac{\sqrt{3}\Gamma(2/3)}{2\pi}\frac{\Gamma(n+1/3)}{\Gamma(n+1)}}\,. \tag{89}$$

6 Equilibrium correlations, diffusion and anomalous diffusion

In this section the momentum correlation function in equilibrium and the time dependence of $\langle x^2(t)\rangle$ are determined.

6.1 Momentum correlation function in equilibrium

The correlation function of momentum in equilibrium is obtained from (49). We have

$$\langle p_t p_0\rangle_{\mathrm{eq.}} = p_0^2 \left(\frac{D_1}{\gamma p_0^2}\right)^{2/3} \sum_n Z_{0n}^2 \exp(\lambda_n^- t')\,. \tag{90}$$

Using (79) we obtain

$$\langle p_t p_0\rangle_{\mathrm{eq.}} = \frac{\Gamma(4/3)}{3^{1/3}\Gamma(5/3)} \left(\frac{p_0 D_1}{\gamma}\right)^{2/3} \mathrm{e}^{-2\gamma t} F_{21}\left(\frac{1}{3}, \frac{1}{3}; \frac{5}{3}; \mathrm{e}^{-3\gamma t}\right) \tag{91}$$

for $t > 0$. Here F_{21} is a hypergeometric function [20]. It follows that $\langle p_t p_0\rangle_{\mathrm{eq.}}$ decays as $\exp(-2\gamma t)$ at large times as opposed to $\exp(-\gamma t)$ in the Ornstein-Uhlenbeck process.

6.2 Diffusion at long times

We now turn to $\langle x^2(t)\rangle$. This expectation value is calculated using eqs. (52), (53), (79), and (89). We have

$$\langle x^2(t)\rangle \;=\; \left(\frac{p_0 D_1}{\gamma}\right)^{2/3} \frac{1}{m^2\gamma^2} \sum_{k=0}^{\infty} \sum_{l=k-1}^{\infty} \frac{\psi_k^+(0)}{\psi_0^+(0)} Z_{0l} Z_{kl} T_{kl} \tag{92}$$

with

$$\begin{aligned} T_{kl}(t') &= \int_0^{t'} \mathrm{d}t_1' \int_{t_1'}^{t'} \mathrm{d}t_2'\, \mathrm{e}^{\lambda_l^-(t_2'-t_1')+\lambda_k^+ t_1'} + \int_0^{t'} \mathrm{d}t_1' \int_0^{t_1'} \mathrm{d}t_2'\, \mathrm{e}^{\lambda_l^-(t_1'-t_2')+\lambda_k^+ t_2'} \\ &= 2\frac{\lambda_k^+(1-\mathrm{e}^{\lambda_l^- t'}) - \lambda_l^-(1-\mathrm{e}^{\lambda_k^+ t'})}{\lambda_l^- \lambda_k^+ (\lambda_k^+ - \lambda_l^-)}\,. \end{aligned} \tag{93}$$

We define

$$A_{kl} \equiv \frac{\psi_k^+(0)}{\psi_0^+(0)} Z_{0l} Z_{kl} = \frac{3^{2/3}\Gamma(2/3)^2}{12\pi^2} \frac{(k+l+1)\Gamma(k+1/3)\Gamma(l+1/3)\Gamma(l-k+1/3)}{\Gamma(k+1)\Gamma(l+5/3)\Gamma(l-k+2)} \tag{94}$$

for $k \geq l-1$ and zero otherwise. In order to determine $\langle x^2(t)\rangle$, the sum

$$S(t') = \sum_{kl} A_{kl} T_{kl}(t') \tag{95}$$

is required. Note that $A_{kl} = 0$ for $k < l-1$. Consider the behaviour of (95) at large values of t'. We write the $k = 0$ term separately

$$T_{0l} = -\frac{2t'}{\lambda_l^-} + \frac{2}{\lambda_l^{-2}}(\mathrm{e}^{\lambda_l^- t'} - 1)\,. \tag{96}$$

This gives

$$
\begin{aligned}
\langle x^2(t)\rangle &= 2\mathcal{D}_x t \\
&+ \left(\frac{p_0 D_1}{\gamma}\right)^{2/3}\frac{1}{m^2\gamma^2}\sum_{l=0}^{\infty} Z_{0l}^2 \frac{2}{\lambda_l^{-2}}(\mathrm{e}^{\lambda_l^- t'}-1) \\
&+ \left(\frac{p_0 D_1}{\gamma}\right)^{2/3}\frac{1}{m^2\gamma^2}\sum_{k=1}^{\infty}\sum_{l=k-1}^{\infty}\frac{\psi_k^+(0)}{\psi_0^+(0)} Z_{0l} Z_{kl} T_{kl}(t')\,.
\end{aligned}
\tag{97}
$$

At large t' the secular term dominates and diffusion is thus recovered. The diffusion constant is obtained as [2]

$$
\mathcal{D}_x = \frac{(p_0 D_1)^{2/3}}{m^2\gamma^{5/3}}\frac{\pi 3^{-5/6}}{2\,\Gamma(2/3)^2} F_{32}\Big(\frac{1}{3},\frac{1}{3},\frac{2}{3};\frac{5}{3},\frac{5}{3};1\Big)\,. \tag{98}
$$

6.3 Anomalous diffusion at short times

Next we consider short times. In order to evaluate (95) at small values of t', we replace the sums in (95) by integrals:

$$
S(t') = \int_0^\infty \mathrm{d}l \int_0^l \mathrm{d}k\ T(k,l,t)A(k,l)\,. \tag{99}
$$

In order to evaluate (99) we use the asymptotic form of the coefficients A_{kl}:

$$
A(k,l) \sim \frac{3^{2/3}\Gamma(2/3)^2}{12\pi^2}\frac{k+l}{k^{2/3}l^{4/3}(l-k)^{5/3}}\,. \tag{100}
$$

The coefficients $A(k,l)$ exhibit non-integrable divergence $k\to l$. In view of this divergence we make use of a sum rule of the A_{kl}:

$$
\sum_{k=0}^{l+1} A_{kl} = 0\,. \tag{101}
$$

It can be derived by considering

$$
\sum_{k=0}^{l+1}\psi_k^+(z)\langle\psi_k^+|\hat{z}|\psi_l^-\rangle = \sum_{k=0}^{\infty}\psi_k^+(z)\langle\psi_k^+|\hat{z}|\psi_l^-\rangle = \sum_{k=0}^{\infty}\langle z|\psi_k^+\rangle\langle\psi_k^+|\hat{z}|\psi_l^-\rangle = \langle z|\hat{z}|\psi_l^-\rangle \tag{102}
$$

which vanishes for $z=0$. Replacing sums by integrals the sum rule amounts to

$$
\int_0^l \mathrm{d}k\ A(k,l) = 0\,. \tag{103}
$$

Eq. (103) allows us to write

$$
S(t') = \int_0^\infty \mathrm{d}l \int_0^l \mathrm{d}k\ \Big[T(k,l;t') - \lim_{k\to l} T(k,l;t')\Big] A(k,l)\,. \tag{104}
$$

We find that the divergence of $A(k,l)$ is reduced to an integrable divergence by the fact that $T(k,l;t) - \lim_{k\to l} T(k,l;t) = O(k-l)$. Approximately, $T(k,l;t)$ is given by

$$T(k,l;t) = -2\,\frac{3k[1-\exp(-3lt)] - 3l[1-\exp(-3kt)]}{27lk(l-k)}\,. \tag{105}$$

Changing the integration variables in (104) to $x = 3lt$ and $xy = 3kt$, we have $k = xy/(3t)$ and $l = x/(3t)$, and

$$A(x,y,t) = \frac{3^{10/3}\Gamma(2/3)^2}{12\pi^2} t^{8/3} x^{-8/3} \frac{1+y}{y^{2/3}(1-y)^{5/3}}. \tag{106}$$

The Jacobian of the transformation is $J = x/(9t^2)$. In the new variables,

$$T(x,y;t) = -2\,\frac{t^2}{x}\left[\frac{a(x)-a(xy)}{1-y}\right] \tag{107}$$

where $a(x) = [1-\exp(-x)]/x$. Furthermore

$$\lim_{y\to 1} T(x,y;t) = -2\frac{t^2}{x}\lim_{y\to 1}\left[\frac{a(x)-a(xy)}{1-y}\right]\,. \tag{108}$$

Using

$$\lim_{y\to 1} T(x,y;t) = -2\frac{t^2}{x}\left[\frac{\partial}{\partial y}a(xy)\right]\Bigg|_{y=1} = -2\frac{t^2}{x}xa'(x) \tag{109}$$

we finally obtain

$$T(k,l;t) - \lim_{k\to l} T(k,l;t) \quad = \quad -2\,\frac{t^2}{x}\left[\frac{a(x)-a(xy)}{1-y} - xa'(x)\right]. \tag{110}$$

This gives

$$S(t) \quad = \quad -C\,t^{8/3}\int_0^\infty \mathrm{d}x\; x^{-8/3}\int_0^1 \mathrm{d}y\;\left[\frac{a(x)-a(xy)}{1-y} - xa'(x)\right]\frac{1+y}{y^{2/3}(1-y)^{5/3}} \tag{111}$$

with $C = 3^{1/3}\Gamma(2/3)^2/(2\pi^2)$. Equation (111) implies anomalous spatial diffusion at short times:

$$\langle x^2(t)\rangle = \mathcal{C}_x\left((p_0 D_1)^{2/3} m^{-2}\right) t^{8/3} \tag{112}$$

with

$$\mathcal{C}_x = -C\int_0^\infty \mathrm{d}x\; x^{-8/3}\int_0^1 \mathrm{d}y\;\left[\frac{a(x)-a(xy)}{1-y} - xa'(x)\right]\frac{1+y}{y^{2/3}(1-y)^{5/3}}\,. \tag{113}$$

This anomalous diffusion is analogous to that described by Golubovic, Feng and Zeng [9] in the case where there is no damping (they considered the case where $\zeta = 3$ in their paper). Our results give the prefactor as well as the scaling behaviour. It is noteworthy that we are able to solve the problem discussed in [9] by solving a more complex set of equations exactly, and taking a limit.

7 WKB analysis

The staggered ladder spectrum discussed in section 4 is surprising, especially in view of the fact that for large quantum number n we expect that the eigenfunctions of the Hamiltonian (23) might be obtained by WKB theory. In this section we show how to obtain the spectrum (29) of $\hat{H}$ using asymptotic Wentzel-Kramers-Brillouin (WKB) analysis. We show how phase shifts associated with the singularity at $z = 0$ of the Hamiltonian (23) are the source of the staggered spectrum. It turns out that the WKB procedure gives rise to the exact eigenvalues.

In dimensionless coordinates, the classical Hamilton function corresponding to (23) is

$$H_{\rm cl} = \frac{1}{2} - \frac{z^3}{4} - p^2/z\,. \tag{114}$$

Solving $H_{\rm cl} = \lambda$ for p we obtain $p(z,\lambda) = \pm\frac{1}{2}\sqrt{(2-4\lambda)z - z^4}$, while the velocity is

$$\dot{z} = \partial H_{\rm cl}/\partial p = -2p/z\,. \tag{115}$$

The classical trajectories are figure-of-eight orbits, illustrated in figure 2.

The WKB wavefunction is of the form

$$f(z) = \left[z/p(z,\lambda)\right]^{1/2} \exp\left(\pm{\rm i}\int^z {\rm d}z\, p(z,\lambda)\right). \tag{116}$$

The phase of the wave function can be determined as follows. We discuss separately the behaviours of the wavefunction at the origin $z_0 = 0$ and in the vicinity of the regular turning point $z_{\rm t.p.} = (2-4\lambda)^{1/3}$. In the latter case, the wavefunction at $z < z_{\rm t.p.}$ connected with the turning point is

$$f_{\rm t.p.}(z) = C_{\rm t.p.}\left[z/p(z,\lambda)\right]^{1/2} \sin\left(\int_z^{z_{\rm t.p.}} {\rm d}z\, p(z,\lambda) + \frac{\pi}{4}\right). \tag{117}$$

Consider now the behaviour of the wavefunction near the origin. The Hamiltonian has a singularity at $z = 0$. We find an exact solution of the equation when $\gamma = 0$. This equation has a continuous spectrum, but we identify solutions $f_+(z)$ and $f_-(z)$ which correspond respectively to even and odd solutions of the full equation for $\gamma > 0$ (with discrete spectrum). Close to $z = 0$, the damping term in the Hamiltonian is negligible, and for $z > 0$ the eigenfunctions resemble solutions of the equation

$$\partial_z z^{-1}\partial_z f(z) = -\Lambda f(z) \tag{118}$$

where $\Lambda(= -\lambda)$ is a positive constant. Write $f = F'$, and find that

$$\frac{\partial}{\partial z}\left(\frac{F'' + \Lambda z F}{z}\right) = 0 \tag{119}$$

so that $F'' + \Lambda z F = Cz$ for some constant C. Thus we find $G(z) = F(z) - C/\Lambda$ satisfies $G'' + z\Lambda G = 0$, which has solution $G(z) = {\rm Ai}(-\Lambda^{1/3}z)$, and a similar solution constructed from ${\rm Bi}(x)$ (here ${\rm Ai}(y)$ and ${\rm Bi}(y)$ are Airy Ai and Bi functions [20]). The general solution is

$$f(z) = A_1{\rm Ai}'(-\Lambda^{1/3}z) + A_2{\rm Bi}'(-\Lambda^{1/3}z). \tag{120}$$

We must find solutions of this form which resemble the behaviour of the eigenfunctions of the equation with $\gamma > 0$ which obey the boundary conditions

$$\frac{\mathrm{d}^2 f_+(0)}{\mathrm{d}z^2} = 0 \quad \text{and} \quad f_-(0) = 0\,. \tag{121}$$

The functions Ai' and Bi' have the following forms in the neighbourhood of $z = 0$

$$\mathrm{Ai}'(y) = c_1\Big(\frac{y^2}{2} + \mathcal{O}(y^5)\Big) - c_2\Big(1 + \frac{y^3}{3} + \mathcal{O}(y^6)\Big) \tag{122}$$

$$\mathrm{Bi}'(y) = \sqrt{3}c_1\Big(\frac{y^2}{2} + \mathcal{O}(y^5)\Big) + \sqrt{3}c_2\Big(1 + \frac{y^3}{3} + \mathcal{O}(y^6)\Big) \tag{123}$$

with $y = -\Lambda^{1/3}z$, $c_1 = 3^{-2/3}/\Gamma(2/3)$, and $c_2 = 3^{-1/3}/\Gamma(1/3)$. So the positive-parity solution corresponds to the choice

$$A_1^+ = -\sqrt{3} \quad \text{and} \quad A_2^+ = 1 \tag{124}$$

while

$$A_1^- = \sqrt{3} \quad \text{and} \quad A_2^- = 1\,. \tag{125}$$

At large values of y the corresponding wave functions are of the form

$$f_\pm(y) \sim (-y)^{1/4} \sin\Big(\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4} \pm \frac{\pi}{3}\Big)\,. \tag{126}$$

Noting that, near $z = 0$, $\int_0^z \mathrm{d}z\, p(z,\lambda) = (2/3)(-y)^{3/2}$ and $(z/p(z,\lambda))^{1/2} \propto (-y)^{1/4}$, the solution coming from the origin is

$$f_{\mathrm{O},\pm}(z) = C_{\mathrm{O},\pm}\big[z/p(z,\lambda)\big]^{1/2} \sin\Big(\int_0^z \mathrm{d}z\, p(z,\lambda) + \frac{\pi}{4} \pm \frac{\pi}{3}\Big)\,. \tag{127}$$

The forms (117) and (127) should be smoothly connected for $0 < z < z_{\mathrm{t.p.}}$. This requires

$$\int_0^{z_{\mathrm{t.p.}}} \mathrm{d}z\, p(z,\lambda) + \frac{\pi}{2} \pm \frac{\pi}{3} = (n+1)\pi, \tag{128}$$

for $n = 0, 1, \ldots$ together with $C_{\mathrm{t.p.}} = (-1)^n C_{\mathrm{O},\pm}$. Writing

$$S(\lambda) = \int_0^{z_{\mathrm{t.p.}}} \mathrm{d}z\, p(z,\lambda) = \frac{\pi}{12}(-4\lambda + 2)\,, \tag{129}$$

we find the quantisation condition

$$S(\lambda_\pm) = \Big(n + \frac{1}{2} \mp \frac{1}{3}\Big)\pi\,. \tag{130}$$

Using (129) the quantisation condition takes the form

$$\lambda = \begin{cases} -3n & \text{even parity} \\ -3n-2 & \text{odd parity} \end{cases} \tag{131}$$

which corresponds exactly to the spectrum (29) obtained by exactly diagonalising $\hat{H}$.

8 Results for other values of ζ

Up to now we have only considered the case of generic random forcing (where the constant D_1 in equation (8) is not zero), corresponding to the case $\zeta = 1$ in equation (9). In this section we explain two cases where other values of ζ arise and briefly describe results for arbitrary positive values of ζ, analogous to the results obtained in section 6.

First consider the case where the force is the gradient of a potential, $f(x,t) = \partial V(x,t)/\partial x$. We assume that the potential has mean value zero and correlation function $\mathcal{C}(X,T) = \langle V(x+X,t+T)V(x,t)\rangle$. Assuming that $\mathcal{C}(X,T)$ is sufficiently differentiable at $T=0$, the diffusion constant is

$$\begin{aligned} D(p) &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}t \, \left\langle \frac{\partial V}{\partial x}(pt/m,t)\frac{\partial V}{\partial x}(0,0)\right\rangle \\ &= \frac{-m}{2|p|}\int_{-\infty}^{\infty} \mathrm{d}X \, \frac{\partial^2 \mathcal{C}}{\partial X^2}(X, mX/p) \\ &= \frac{-m}{2|p|}\int_{-\infty}^{\infty} \mathrm{d}X \, \left[\frac{\partial^2 \mathcal{C}}{\partial x^2}(X,0) + \frac{m}{p}X\frac{\partial^3 \mathcal{C}}{\partial^2 X \partial T}(X,0) \right. \\ &\qquad \left. + \frac{m^2}{2p^2}X^2\frac{\partial^4 \mathcal{C}}{\partial^2 X\partial^2 T}(X,0) + O(X^3)\right] \, . \end{aligned} \tag{132}$$

Integration by parts shows that the integral over the first term of the expansion is zero, and the integral over the second term is zero by symmetry. The leading-order contribution in $|p|^{-1}$ comes from the third term. Integrating this term by parts twice gives

$$D(p) \sim \frac{-m^3}{2|p|^3}\int_{-\infty}^{\infty} \mathrm{d}X \, \frac{\partial^2 \mathcal{C}}{\partial T^2}(X,0) \, . \tag{133}$$

Thus in the case of a potential force with a sufficiently smooth correlation function we have $\zeta = 3$.

An exceptional case which is worthy of comment is when the potential $V(x,t)$ is itself generated from a set of Ornstein-Uhlenbeck processes $A_j(t)$ by writing $V(x,t) = \sum_j A_j(t)\Phi_j(x)$, where the $\Phi_j(x)$ are elements of some suitable set of basis functions. In this case the correlation function of $V(x,t)$ is of the form $c(x)\exp(-\gamma|t|)$ [for some function $c(x)$]. Then the second term in the expansion on the final line of equation (132) does not vanish by symmetry and we find $D(p) \propto |p|^{-2}$, that is $\zeta = 2$.

For general positive values of ζ the Hamiltonian (22) is replaced by

$$\hat{H} = \frac{1}{2} - \frac{1}{4}|z|^{2+\zeta} + \frac{\partial}{\partial z}\frac{1}{|z|^\zeta}\frac{\partial}{\partial z} \, . \tag{134}$$

Its ground state

$$\lambda_0^+ = 0 \quad \text{and} \quad \psi_0^+(z) = \mathcal{C}_0^+ \mathrm{e}^{-|z|^{\zeta+2}/(4+2\zeta)} \tag{135}$$

and first excited state

$$\lambda_0^- = -1-\zeta \quad \text{and} \quad \psi_0^-(z) = \mathcal{C}_0^- z|z|^\zeta \, \mathrm{e}^{-|z|^{\zeta+2}/(4+2\zeta)} \tag{136}$$

are found by inspection. Raising and lowering operators can be introduced in a manner analogous to eqs. (24-26). We write

$$\hat{H} = \hat{a}^-|z|^{-\zeta}\hat{a}^+ \tag{137}$$

with $\hat{a}^{\pm} = \partial_z \pm z|z|^{\zeta}/2$. The operators

$$\hat{A} = \hat{a}^{+}|z|^{-\zeta}\hat{a}^{+} \quad \text{and} \quad \hat{A}^{+} = \hat{a}^{-}|z|^{-\zeta}\hat{a}^{-} \tag{138}$$

satisfy

$$[\hat{H}, \hat{A}] = (2+\zeta)\hat{A} \quad \text{and} \quad [\hat{H}, \hat{A}^{+}] = -(2+\zeta)\hat{A}^{+}\,. \tag{139}$$

and act as lowering and raising operators. For the spectrum of $\hat{H}$ we obtain

$$\lambda_n^{+} = -(2+\zeta)\,n \quad \text{and} \quad \lambda_n^{-} = -(2+\zeta)n - 1 - \zeta\,. \tag{140}$$

These expressions replace (29). Note also that the commutator of $\hat{A}$ and $\hat{A}^{+}$ is

$$[\hat{A}, \hat{A}^{+}] = -(2+\zeta)(\hat{H} + \hat{G}) \tag{141}$$

where $\hat{G} = \hat{a}^{+}|z|^{-\zeta}\hat{a}^{-}$ and $\hat{H} - \hat{G} = \hat{I}$. The normalisation of the eigenstates

$$\hat{A}^{+}|\psi_n^{-}\rangle = C_{n+1}^{-}|\psi_{n+1}^{-}\rangle \tag{142}$$

$$\hat{A}|\psi_n^{-}\rangle = C_n^{-}\;|\psi_{n-1}^{-}\rangle \tag{143}$$

is determined as in section 4.1.2. We obtain

$$(C_{n+1}^{-})^2 = (2+\zeta)(n+1)[(2+\zeta)n + 3 + 2\zeta] \tag{144}$$

and

$$(C_{n+1}^{+})^2 = (2+\zeta)(n+1)[(2+\zeta)n + 1]\,. \tag{145}$$

The results of section 5 for the matrix elements $Z_{mn} = \langle\psi_m^{-}|\hat{z}|\psi_n\rangle$ and for $\psi_n^{+}(0)/\psi_0^{+}(0)$ generalise as follows:

$$Z_{mn} = (-1)^{n-m}\frac{(2+\zeta)^{-\frac{1+\zeta}{2+\zeta}}}{\Gamma(\frac{\zeta}{2+\zeta})}(m+n+1)\frac{\Gamma(\frac{\zeta}{2+\zeta} - m + n)\sqrt{\Gamma(n+1)\Gamma(\frac{1}{2+\zeta} + m)}}{\Gamma(2-m+n)\sqrt{\Gamma(\frac{3+2\zeta}{2+\zeta} + n)\Gamma(m+1)}}\,, \tag{146}$$

$$\psi_n^{+}(0)/\psi_0^{+}(0) = (-1)^n\sqrt{\frac{\Gamma[((2+\zeta)n+1)/(2+\zeta)]}{\Gamma(n+1)\Gamma(1/(2+\zeta))}}\,. \tag{147}$$

This allows us to obtain, for example, the diffusion constant

$$\mathcal{D}_x = \frac{1}{m^2}\left(\frac{p_0^{2\zeta}D_\zeta^2}{\gamma^{4+\zeta}}\right)^{\frac{1}{2+\zeta}}\frac{(2+\zeta)^{-\frac{4+3\zeta}{2+\zeta}}\pi F_{32}(\frac{\zeta}{2+\zeta}, \frac{\zeta}{2+\zeta}, \frac{1+\zeta}{2+\zeta}; \frac{3+2\zeta}{2+\zeta}, \frac{3+2\zeta}{2+\zeta}; 1)}{\sin(\frac{\pi}{2+\zeta})\Gamma(\frac{3+2\zeta}{2+\zeta})^2} \tag{148}$$

describing the dynamics at large times. Upon substituting $\zeta = 0$, eq. (148) reproduces the standard Ornstein-Uhlenbeck result, and it gives (98) for $\zeta = 1$.

For the short-time anomalous diffusion we obtain

$$\langle x^2(t)\rangle = \mathcal{C}_x((p_0^{\zeta}D_\zeta)^{\frac{2}{2+\zeta}}m^{-2})t^{\frac{6+2\zeta}{2+\zeta}} \tag{149}$$

with

$$\mathcal{C}_x = -C\int_0^\infty \mathrm{d}x\, x^{-\frac{6+2\zeta}{2+\zeta}} \int_0^1 \mathrm{d}y \left[\frac{a(x)-a(xy)}{1-y} - xa'(x)\right] \frac{1+y}{y^{\frac{1+\zeta}{2+\zeta}}(1-y)^{\frac{4+\zeta}{2+\zeta}}}, \tag{150}$$

where $a(x)$ is the same as in (113), and

$$C = \frac{2(2+\zeta)^{-\frac{2\zeta}{2+\zeta}}}{\Gamma(\frac{\zeta}{2+\zeta})^2}. \tag{151}$$

This reproduces (112) for $\zeta = 1$ and concludes our summary of results for other than generic random forcing.

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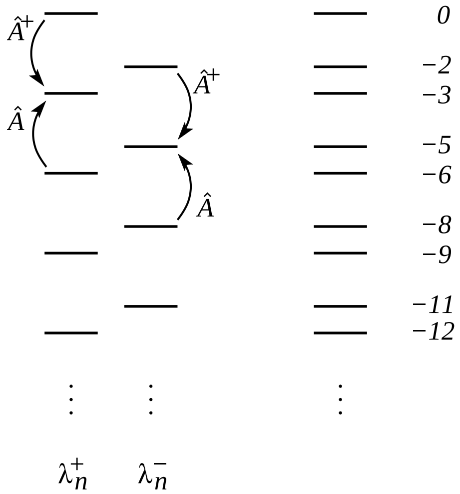

Figure 1: The spectrum of $\hat{H}$ consists of two equally spaced (ladder) spectra λ_n^- and λ_n^+ which are 'staggered' (that is, they are interleaved with un-even spacings). $\hat{A}$ and $\hat{A}^+$ do not change the parity of the eigenfunctions.

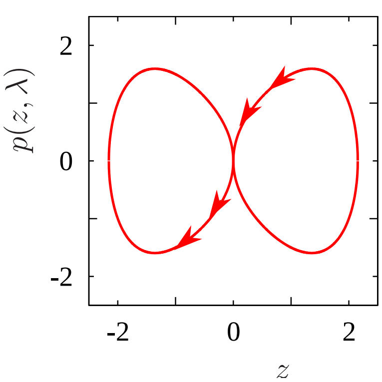

Figure 2: The trajectories of the classical Hamiltonian (114) are figure-of-eight orbits.